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Topology and its Applications 68 (1996) 67–81

TOPOLOGY AND ITS APPLICATIONS

Generalized Lefschetz numbers of pushout maps

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Received 6 October 1994; revised 6 February 1995

Abstract

There is a useful way of defining a new map f from other given maps f_A, f_1 and f_2: the pushout construction. The pushout map f is analogous to the well known pushout of topological spaces. In this paper we prove a Pushout formula relating the generalized Lefschetz number of the pushout map f to those of the given maps f_A, f_1 and f_2. This provides a tool to compute generalized Lefschetz numbers and Nielsen numbers in a rather easy way. Some interesting examples are given at the end of the paper.

Keywords: Nielsen number; Generalized Lefschetz number; Pushout; Fixed point; Nielsen fixed point theory

AMS classification: Primary 55M20, Secondary 55P99

1. Introduction

Let A, X_1 and X_2 be finite connected CW-complexes. Suppose that A is a subcomplex of X_1 and that $i_2 : A \mapsto X_2$ is a given cellular map. Let X be the space obtained by attaching X_1 to X_2 via i_2 (X is the pushout space). It is well known that if we have self maps $f_A : A \mapsto A$, $f_1 : X_1 \mapsto X_1$ and $f_2 : X_2 \mapsto X_2$ such that $f_1 i_1 = i_1 f_A$ and $f_2 i_2 = i_2 f_A$, where $i_1 : A \mapsto X_1$ is the inclusion, then we can define the pushout map $f : X \mapsto X$.

The problem is to compute the generalized Lefschetz number $\mathcal{L}(f)$ of f. It is natural to expect that $\mathcal{L}(f)$ is related to $\mathcal{L}(f_A)$, $\mathcal{L}(f_1)$ and $\mathcal{L}(f_2)$. The main result of the paper is the Pushout formula; this is an easy relation given in Theorem 3.2.1.

In Section 2 we give the necessary background to understand the problem and to prepare us for the proof of the formula. In Section 2.1 we give a straightforward definition of the generalized Lefschetz numbers; there we also define some functions to be used

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SSDI 0166-8641(96)00040-2

later on. In Section 2.2 we give a proof of the well known homotopy invariance of $\mathcal{L}$ which allows us to introduce the function H_* induced on the fixed point classes by a homotopy H. In Section 2.3 we quote the Lefschetz trace formula for $\mathcal{L}$, given in [3]; this is the key-fact for the proof of the Pushout formula.

In Section 3 we state the problem, the pushout construction with full details, and the Pushout formula. In Section 3.3 we show some particular cases in which the formula may be used. They come from the most elementary pushout constructions of topological spaces.

The proof of the formula is given in Section 4. In order to simplify it, we split it up in three lemmas. At last, in Section 5, several examples are given. The Pushout formula really allows us to easily compute generalized Lefschetz numbers of these maps. They are interesting by themselves because they are simple counter-examples to the Lefschetz fixed point theorem, as the classical one by McCord (cf. [5]) for a wider class of spaces.

Sincere thanks are due to Renzo Piccinini and Robert Freeman Brown for their invaluable help and encouragement.

2. Preliminaries

2.1. Generalized Lefschetz numbers

Let X be a finite connected CW-complex. We say that a self map $f : X \mapsto X$ is *path-based* if a base path $w : I \mapsto X$ has been chosen so that $f(w(0)) = w(1)$. We denote with (f, w) a path-based map f with base path w. Let $\mathcal{AM}_{\text{pb}}$ be the category whose objects are all such path-based self maps. For all given objects $(f, w) : X \mapsto X$ and $(g, v) : Y \mapsto Y$, the morphisms are maps $h : X \mapsto Y$ such that $gh = hf$ and $h(w) = v$; we use the symbol h to denote both the morphism $h : (f, w) \mapsto (g, v)$ and the map $h : X \mapsto Y$.

There is an interesting functor acting on $\mathcal{AM}_{\text{pb}}$: the Reidemeister functor $\mathcal{R} : \mathcal{AM}_{\text{pb}} \mapsto$ *Set*. For every path-based self map (f, w), with $x_0 := w(0)$, let $\pi(f, w)$ be the endomorphism $\pi(f, w) : \pi_1(X, x_0) \mapsto \pi_1(X, x_0)$ defined by $\pi(f, w)(\alpha) := wf(\alpha)w^{-1}$ for all $\alpha \in \pi_1(X, x_0)$. For any fixed (f, w), $\pi_1(X, x_0)$ acts on itself if we set $g \cdot x := gx\pi(f, w)(g^{-1})$ for every $g, x \in \pi_1(X, x_0)$. We call the orbit set the *Reidemeister set* of (f, w), written as $\mathcal{R}(f, w)$. The orbit of the element $g \in \pi_1(X, x_0)$ is indicated by $[g] \in \mathcal{R}(f, w)$. If we have a morphism $h : (f, w) \mapsto (g, h(w))$ of $\mathcal{AM}_{\text{pb}}$, we can define $\mathcal{R}(h) := h_* : \mathcal{R}(f, w) \mapsto \mathcal{R}(g, h(w))$ setting $h_*([g]) := [h(g)]$ for every $g \in \pi_1(X, x_0)$. The Reidemeister functor is analogous to the fixed point class functor of [4]. At last, there is a trivial functor $\mathbf{Z} : Set \mapsto Ab$ from the category of sets to the category of (free) abelian groups: to every set S we associate the free abelian group $\mathbf{Z}S$ generated by S, and to every map $m : S_1 \mapsto S_2$, the induced homomorphism $m : \mathbf{Z}S_1 \mapsto \mathbf{Z}S_2$. Note that m may denote either the map on S_1 or the homomorphism on $\mathbf{Z}S_1$.

In this framework it is easy to define the *generalized Lefschetz number* of a self map f. Take a base path w for f, and define the *coordinate* function $\mathrm{cd}_w : \mathrm{Fix}(f) \mapsto \mathcal{R}(f,w)$ from the set of fixed points $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ to the Reidemeister orbits of (f,w): for every $x \in \mathrm{Fix}(f)$ choose a path $\lambda_x : I \mapsto X$ such that $\lambda_x(0) = x_0$ and $\lambda_x(1) = x$, and let $\mathrm{cd}_w(x) := [\lambda_x f(\lambda_x^{-1}) w^{-1}]$. The coordinate function is well defined and can distinguish different fixed point classes: two fixed points x and y of f are in the same fixed point class if and only if $\mathrm{cd}_w(x) = \mathrm{cd}_w(y)$ (cf. [1,4]). Hence we can define the index $\mathrm{ind}(\xi)$ of an element ξ of $\mathcal{R}(f,w)$ letting it to be equal to the fixed point index $\mathrm{ind}(f, \mathrm{cd}_w^{-1}(\xi))$ of the corresponding fixed point class $\mathrm{cd}_w^{-1}(\xi)$, as in [1,4]. If the class is empty, the index is naturally zero. The generalized Lefschetz number of the self map (f,w)

$$\mathcal{L}(f,w) := \sum_{\xi \in \mathcal{R}(f,w)} \mathrm{Ind}(\xi) \cdot \xi$$

is actually an element of $\mathbf{Z}\mathcal{R}(f,w)$. The reader should recall that the sum of the indices $\mathrm{ind}(\xi)$ coincides with the classical Lefschetz number and that the number of elements $\xi \in \mathcal{R}(f,w)$ having nonzero index is the Nielsen number of the map f. For all the important properties of these numbers and their very close relations with fixed point theory, see, e.g., [4,1,3].

2.2. Homotopy invariance

One question easily arises: what happens if we take different base paths for a self map f? Let f be given, and take two base paths w and w'. For every path λ such that $\lambda(0) = w(0)$ and $\lambda(1) = w'(0)$ we can define a change of coordinate $\lambda_* : \mathcal{R}(f,w) \mapsto \mathcal{R}(f,w')$ if we put $\lambda_*([\alpha]) := [\lambda^{-1}\alpha w f(\lambda) {w'}^{-1}]$ for every $\alpha \in \pi_1(X, w(0))$. This is an index-preserving bijection because the equality $\mathrm{cd}_{w'} = \lambda_* \, \mathrm{cd}_w$ holds true.

The homotopy invariance of the generalized Lefschetz number is a classical well known result; the proof of this fact is based on the trace formula and chain homotopy properties. In this paper we offer a direct proof of the homotopy invariance, following a slightly different route. Let (f,w) and (g,v) be two homotopic self maps, with base paths w and v with the same starting point $w(0) = v(0) = x_0$. For every homotopy $H : f \sim g$, let the path defined by $\gamma_H(t) := H(x_0, t)$ $\forall t \in I$, be the track of the homotopy H on x_0.

Proposition 2.2.1 (Homotopy invariance). *If we define*

$$H_*([\alpha]) := \left[\alpha w \gamma_H v^{-1}\right]$$

for every $[\alpha] \in \mathcal{R}(f,w)$*,we obtain an index preserving bijection*

$$H_* : \mathcal{R}(f,w) \mapsto \mathcal{R}(g,w).$$

Consequently H_* *is such that* $\mathcal{L}(g,v) = H_* \mathcal{L}(f,w)$.

Proof. Let $\overline{H}: X \times I \mapsto X \times I$ be the fat homotopy of H, defined by $\overline{H}(x,t) := (H(x,t),t)$ for every $(x,t) \in X \times I$. Let $i_0: X \mapsto X \times I$ and $i_1: X \mapsto X \times I$ be the inclusions $i_0(x) := (x,0)$ and $i_1(x) := (x,1)$. As morphisms of $\mathcal{AM}_{\mathrm{pb}}$, we can write

$$i_0: (f,w) \mapsto (\overline{H}, i_0(w)) \quad \text{and} \quad i_1: (g,v) \mapsto (\overline{H}, i_1(v)).$$

Following [1,4], it is trivial to show that

$$i_{0*}: \mathcal{R}(f,w) \mapsto \mathcal{R}(\overline{H}, i_0(w)) \quad \text{and} \quad i_{1*}: \mathcal{R}(g,v) \mapsto \mathcal{R}(\overline{H}, i_1(v))$$

are index preserving bijections. If we take the vertical path c in $X \times I$ defined by $c(t) := (x_0, t)$ for all $t \in I$, the induced change of coordinate $c_*: \mathcal{R}(\overline{H}, i_0(w)) \mapsto \mathcal{R}(\overline{H}, i_1(v))$ is an index preserving bijection. Now we can put $H_* := i_{1*}^{-1} c_* i_{0*}$. We have an obviously index preserving bijection which coincides with that of the proposition. □

2.3. *The Lefschetz–Husseini trace formula*

Let $\widetilde{X}$ be the universal covering space of X. Note that $\widetilde{X}$ is not necessarily a finite CW-complex. For every integer $q \geqslant 0$, let $C_q(\widetilde{X})$ denote the qth cellular chain group $C_q(\widetilde{X}) := H_q(\widetilde{X}^{(q)}, \widetilde{X}^{(q-1)}; \mathbf{Z})$. It is a right finitely generated free $\mathbf{Z}[\pi_1(X)]$-module (cf. [3]). Let $(f,w): X \mapsto X$ be a path-based cellular self map, with $w(0) = x_0$. We can view $\widetilde{X}$ as the set of all homotopy classes rel. endpoints of paths in X starting at x_0, and define a canonical cellular lifting of (f,w), $\tilde{f}: \widetilde{X} \mapsto \widetilde{X}$, setting $\tilde{f}([\lambda]) := [wf(\lambda)]$. Hence, there is a canonical module homomorphism $C_q(\tilde{f}): C_q(\widetilde{X}) \mapsto C_q(\widetilde{X})$ which is also a $(\pi_1(X,x_0), f_*)$-homomorphism (cf. [3]), where f_* is the induced endomorphism on $\pi_1(X,x_0)$. Thus the Stallings trace of $C_q(\tilde{f})$ can be defined: it is the element of $\mathbf{Z}\mathcal{R}(f,w)$ given by the sum of the diagonal terms of the matrix representing $C_q(\tilde{f})$ in any free $\mathbf{Z}\pi_1(X)$ basis for $C_q(\widetilde{X})$, modulo Reidemeister action. It will be denoted by $\mathrm{Tr}_{f_*}[C_q(\tilde{f})]$. We are now in the position to state the Lefschetz–Husseini trace formula.

$$\mathcal{L}(f,w) = \sum_{q \geqslant 0} (-1)^q \, \mathrm{Tr}_{f_*} \left[C_q(\tilde{f})\right].$$

3. The Pushout formula

3.1. *Pushouts in $\mathcal{AM}_{\mathrm{pb}}$*

Let us take the self maps $f_A: A \mapsto A$, $f_1: X_1 \mapsto X_1$ and $f_2: X_2 \mapsto X_2$. For any couple of maps $i_1: A \mapsto X_1$ and $i_2: A \mapsto X_2$ such that $i_1 f_A = f_1 i_1$ and $f_2 i_2 = i_2 f_A$, denote the pushout space of X_1 and X_2 via i_1 and i_2 with $X := X_1 \sqcup_A X_2$; we also denote with $j_1: X_1 \mapsto X$ and $j_2: X_2 \mapsto X$ the two induced maps. We see that there exists a unique continuous self map $f: X \mapsto X$ such that $f j_1 = j_1 f_1$ and $f j_2 = j_2 f_2$. The map f, which will also be indicated by $f_1 \sqcup_{f_A} f_2$, is the Pushout map of f_1 and f_2 via i_1 and i_2. This is a kind of pushout construction in the category of all self maps (similar to $\mathcal{AM}_{\rm pb}$). It satisfies the universal property of colimits in this category and satisfies the horizontal and vertical composition laws (cf. [6]).

$$
\begin{tikzcd}
A \arrow[rrr, "i_2"] \arrow[ddd, "i_1"'] & & & X_2 \arrow[ddd, "j_2"] \\
& A \arrow[ul, "f_A"'] \arrow[r, "i_2"] \arrow[d, "i_1"'] & X_2 \arrow[ur, "f_2"] \arrow[d, "j_2"] & \\
& X_1 \arrow[dl, "f_1"'] \arrow[r, "j_1"] & X \arrow[dr, "f"] & \\
X_1 \arrow[rrr, "j_1"] & & & X
\end{tikzcd}
$$

Now let us suppose f_A, f_1, f_2, i_1 and i_2 are cellular maps on finite connected CW-complexes A, X_1 and X_2 as before. Then X is a finite CW-complex and the maps j_1, j_2, f are cellular. Hence this construction is possible in $\mathcal{AM}_{\rm pb}$ whenever the starting maps are cellular. Please note that for the previous construction it suffices to choose a base path w for f_A and take $i_1(w)$, $i_2(w)$ and $j_1 i_1(w)$ as base paths for X_1, X_2 and X. The previous diagram will be called a pushout diagram in $\mathcal{AM}_{\rm pb}$.

3.2. *The generalized Lefschetz number of a pushout map*

We can think at the previous diagram like a square diagram in $\mathcal{AM}_{\rm pb}$ with base paths as explained before. Assume that we know the generalized Lefschetz numbers of f_A, f_1 and f_2. Is it possible to say something about the Lefschetz number of f? Is it related to them? Remember that all the involved maps are cellular. Let us take the image of the square diagram by the $\mathbf{Z}\mathcal{R}$ functor: we get a square diagram of free abelian groups,

$$
\begin{tikzcd}
\mathbf{Z}\mathcal{R}(f_A, w) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"'] & \mathbf{Z}\mathcal{R}(f_2, i_2(w)) \arrow[d, "j_{2*}"] \\
\mathbf{Z}\mathcal{R}(f_1, i_1(w)) \arrow[r, "j_{1*}"] & \mathbf{Z}\mathcal{R}(f, j_1 i_1(w))
\end{tikzcd}
$$

in which the known $\mathcal{L}(f_A, w)$, $\mathcal{L}(f_1, i_1(w))$ and $\mathcal{L}(f_2, i_2(w))$ belong to their respective free abelian groups. In what follows, we omit the base paths which are supposed to be fixed once and for all.

Theorem 3.2.1 (Pushout formula). *If i_1 is an inclusion, then*

$$\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}\mathcal{L}(f_2) - j_{1*}i_{1*}\mathcal{L}(f_A).$$

Clearly, this allows us to compute $\mathcal{L}(f)$ once we know the three terms of the right hand side of the equation. Sometimes this turns out to be a straightforward way to compute generalized Lefschetz numbers and Nielsen numbers of maps.

3.3. Some particular cases

We know that the pushout construction is a general method of getting new topological spaces from some given ones. The most common examples in this direction are the union of subspaces, the quotient space, the one-point union (wedge) and the connected sum (or the adjunction of cells). In almost all these cases the Pushout formula becomes simpler.

Union spaces

Let $X = X_1 \cup X_2$ be the union of two connected sub-complexes, such that the intersection $A = X_1 \cap X_2$ is connected. For every cellular maps $f_1 : X_1 \mapsto X_1$ and $f_2 : X_2 \mapsto X_2$ that coincide on A, let f be the extended map $f : X \mapsto X$. If we know $\mathcal{L}(f_A)$, $\mathcal{L}(f_1)$ and $\mathcal{L}(f_2)$ the Pushout formula yields $\mathcal{L}(f)$.

Quotient spaces

Let X be a finite connected CW-complex, f a cellular self map of X and A a subcomplex of X such that $f(A) \subseteq A$. Let $q : X \mapsto X/A$ be the quotient map, $\bar{f}$ the induced self map of X/A and f_A the self map restricted to A. Then

$$\mathcal{L}(\bar{f}) = q_* \mathcal{L}(f) + \bigl(1 - L(f_A)\bigr)[1],$$

where $L(f_A)$ is the classical Lefschetz number of f_A. The proof of this formula is based upon the pushout construction of the quotient space X/A

$$\begin{array}{ccc} A & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow{q} & X/A \end{array}$$

and the fact that $q_* i_* \mathcal{L}(f_A) = L(f_A) \cdot [1]$.

One-point unions

Let $f_1 : (X_1, x_1) \mapsto (X_1, x_1)$ and $f_2 : (X_2, x_2) \mapsto (X_2, x_2)$ be two pointed maps. Then the induced map on the wedge product of X_1 and X_2 satisfies the equation

$$\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}\mathcal{L}(f_2) - [1]$$

where $j_1 : X_1 \mapsto X_1 \bigvee X_2$ and $j_2 : X_2 \mapsto X_1 \bigvee X_2$ are the inclusions on the first and the second factor respectively.

Connected sums

Let M_1 and M_2 be two connected compact triangulated n-manifolds. We can view the connected sum $M_1 \# M_2$ as the pushout space of the manifolds minus open ball D^n along the boundary $A := \partial D^n$ of D^n, $X_1 := M_1 - D^n$ and $X_2 := M_2 - D^n$. If the maps $f_1 : X_1 \mapsto X_1$ and $f_2 : X_2 \mapsto X_2$ are suitably defined on A, we can get a whole map $f : M_1 \# M_2 \mapsto M_1 \# M_2$. For every $n \geqslant 2$ the formula holds, and for $n \geqslant 3$, because S^n is simply connected, it becomes

$$\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}\mathcal{L}(f_2) - L(f_A)[1],$$

where j_1 and j_2 are the obvious inclusions. Let us remark that in this and in the previous case the only term in the sum that may disappear is the $[1]$ term. Moreover, j_{1*} and j_{2*} are injections. Consequently, the Nielsen number of f may be either $N(f_1)+N(f_2)-1$ or $N(f_1)+N(f_2)-2$. We can select the appropriate case by computing the classical Lefschetz number of f. This happens every time A is a simply connected space.

4. Proof of the formula

Let $M(i_2)$ be the mapping cylinder of the cellular map i_2. Define $M(i_2)=X_2\sqcup_A(A\times I)$ to be the pushout space of X_2 and $A\times I$ via the inclusion $A\mapsto A\times 0\subseteq A\times I$ and the map $i_2\colon A\mapsto X_2$; we note that $M(i_2)$ is a finite connected CW-complex. We can define a cellular inclusion (therefore a cofibration) $ii_2\colon A\mapsto M(i_2)$ setting $ii_2(a):=\bar{\imath}_2(a,1)$, where $\bar{\imath}_2\colon A\times I\mapsto M(i_2)$ is the map induced by the pushout construction of $M(i_2)$; moreover, the map $p\colon M(i_2)\mapsto X_2$ defined by $p_{A\times I}(a,t)=i_2(a)$ and $p_{X_2}(x_2)=x_2$ for every $a\in A$, $t\in I$ and $x_2\in X_2$ is a homotopy equivalence. The equality $i_2=p\,ii_2$ trivially holds true, and the map $\bar{\jmath}\colon X_2\mapsto M(i_2)$ given by the inclusion on the pushout space is the homotopy inverse of p. Actually, $p\bar{\jmath}=1_{X_2}$ and $\bar{\jmath}p\sim 1_{M(i_2)}$ through the homotopy $H\colon M(i_2)\times I\mapsto M(i_2)$ defined by $H(\bar{\imath}_2(a,t),s)=\bar{\imath}_2(a,st)$ for all $(a,t)\in A\times I$, and $H(\bar{\jmath}(x_2),s)=\bar{\jmath}(x_2)$ for all $x_2\in X_2$. All these facts about the mapping cylinder of i_2 are well known. The next step is to bring these concepts into the category $\mathcal{AM}_{\mathrm{pb}}$, by simply defining an analogue of the mapping cylinder in $\mathcal{AM}_{\mathrm{pb}}$ (with a trivial choice of base paths). Just define the map $f_2'\colon M(i_2)\mapsto M(i_2)$ to be $f_2'(\bar{\imath}_2(a,t)):=\bar{\imath}_2(f_A(a),t)$ for all $(a,t)\in A\times I$, and $f_2'(\bar{\jmath}(x_2)):=\bar{\jmath}(f_2(x_2))$ for all $x_2\in X_2$. The map is such that $f_2'\,ii_2=ii_2\,f_A$, and $f_2\,p=p\,f_2'$. It is a cellular map. Therefore we have in $\mathcal{AM}_{\mathrm{pb}}$ an object f_2', a cellular inclusion $ii_2\colon f_A\mapsto f_2'$, and a cellular homotopy equivalence $p\colon f_2'\mapsto f_2$ such that $p\,ii_2=i_2$.

Now define the pushout map of f_1 and f_2' via the cellular inclusions $i_1\colon f_A\mapsto f_1$ and $ii_2\colon f_A\mapsto f_2'$, and call it $f_1\sqcup_{f_A}f_2'$. Let $\bar{\imath}_1\colon f_2'\mapsto f_1\sqcup_{f_A}f_2'$ and $\bar{\imath\imath}_2\colon f_1\mapsto f_1\sqcup_{f_A}f_2'$ be the induced pushout cellular inclusions. It is trivial to define a cellular map $\bar{p}\colon f_1\sqcup_{f_A}f_2'\mapsto f$ such that $\bar{p}\,\bar{\imath\imath}_2=j_1$ and $\bar{p}\,\bar{\imath}_1=j_2\,p$. In other words, the following diagram is commutative in $\mathcal{AM}_{\mathrm{pb}}$, all the maps are cellular, and all the three possible square diagrams are pushout diagrams (by the horizontal composition law).

$$\begin{tikzcd}
f_A \arrow[r, "ii_2"] \arrow[d, "i_1"'] & f_2' \arrow[r, "p"] \arrow[d, "\bar{\imath}_1"'] & f_2 \arrow[d, "j_2"'] \\
f_1 \arrow[r, "\bar{\imath\imath}_2"'] & f_1\sqcup_{f_A} f_2' \arrow[r, "\bar{p}"] & f
\end{tikzcd}$$

We wish to emphasize that both i_1 and ii_2 are inclusions. The base paths are assumed to be chosen in a natural way from a fixed base path w for f_A. The proof of the Pushout formula will follow easily from the next lemmas.

Lemma 4.1. *In the notation of Section* 3.1, *if both* i_1 *and* i_2 *are inclusions, then the formula*

$$\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}\mathcal{L}(f_2) - j_{1*}i_{1*}\mathcal{L}(f_A)$$

holds true, where $f = f_1 \sqcup_{f_A} f_2$.

Proof. Let $\tilde{A}$, $\widetilde{X}_1$, $\widetilde{X}_2$ and $\widetilde{X}$ be the universal covering spaces of A, X_1, X_2 and X. Let $\tilde{f_A}$, $\tilde{f_1}$, $\tilde{f_2}$ and $\tilde{f}$ be the canonical liftings of f_A, f_1, f_2 and f respectively, once we have set the base paths w, $i_1(w)$, $i_2(w)$ and $j_1 i_1(w)$. Let $\tilde{i_1}$, $\tilde{i_2}$, $\tilde{j_1}$ and $\tilde{j_2}$ be the canonical liftings of i_1, i_2, j_1 and j_2; for example, $\tilde{i_1}$ is defined by $\tilde{i_1}([\lambda]) := [i_1(\lambda)]$ for every $[\lambda] \in \tilde{A}$ (cf. Section 2.3). We can draw a covering diagram of the diagram in Section 3.1, at a universal covering space level.

Let q be any positive integer. As in Section 2.3, we can assume $C_q(\widetilde{X})$ to be a free finitely generated right $\pi_1(X)$-module; at the same time, $C_q(\tilde{A})$, $C_q(\widetilde{X}_1)$ and $C_q(\widetilde{X}_2)$ are respectively, a free finitely generated right $\pi_1(A)$-module, a $\pi_1(X_1)$-module and a $\pi_1(X_2)$-module. Their bases may consist of sets of cells of the respective covering spaces, obtained by lifting the cells of the base spaces. More precisely, in the case of A, let $p:\tilde{A} \mapsto A$ be the covering map; for any q-cell e^q of A, $p^{-1}(e^q)$ is the disjoint union of q-cells in $\tilde{A}$. We can pick one, and consider the corresponding element in $C_q(\tilde{A})$. If we proceed in the same way, we get a free $\mathbf{Z}\pi_1(A)$-basis for $C_q(\tilde{A})$. Let $a_1, a_2, \ldots, a_k \in C_q(\tilde{A})$ be the elements of such a basis. By the injectivity of i_1, we can take in $C_q(\widetilde{X}_1)$ a free $\mathbf{Z}\pi_1(X_1)$-basis, whose elements are $b_1, b_2, \ldots, b_k, \ldots, b_{k+s}$, such that for every $u = 1, \ldots, k$, $b_u = C_q(\tilde{i_1})(a_u)$. In the same manner, there exists a free $\mathbf{Z}\pi_1(X_2)$-basis in $C_q(\widetilde{X}_2)$, consisting of the elements $c_1, c_2, \ldots, c_k, \ldots, c_{k+t}$, such that for every $u = 1, \ldots, k$, $c_u = C_q(\tilde{i_2})(a_u)$. Both these bases are made up by lifting the q-cells of X_1 and X_2. Therefore, all the elements $C_q(\tilde{j_1})(b_u)$ with $u = 1, \ldots, k+s$ and $C_q(\tilde{j_2})(c_v)$ with $v = 1, \ldots, k+t$ form a free $\mathbf{Z}\pi_1(X)$-basis for $C_q(\widetilde{X})$.

Using these bases, we find the entries of the matrices representing $C_q(\tilde{f_A})$, $C_q(\tilde{f_1})$, $C_q(\tilde{f_2})$ and $C_q(\tilde{f})$. They are elements of $\mathbf{Z}\pi_1(A)$ such that

$$C_q(\tilde{f_A})(a_u) = \sum_{v=1}^{k} a_v A_u^v$$

for all $u = 1, \ldots, k$; elements of $\mathbf{Z}\pi_1(X_1)$ such that

$$C_q(\tilde{f_1})(b_u) = \sum_{v=1}^{k+s} b_v B_u^v$$

for all $u = 1, \ldots, k+s$; elements of $\mathbf{Z}\pi_1(X_2)$ such that

$$C_q(\tilde{f_2})(c_u) = \sum_{v=1}^{k+t} c_v C_u^v$$

for all $u = 1, \ldots, k+t$. Hence, for all $u = 1, \ldots, k+s$,

$$C_q(\tilde{f})\big(C_q(\tilde{j_1})(b_u)\big) = C_q(\tilde{j_1})\left(\sum_{v=1}^{k+s} b_v B_u^v\right),$$

and for all $u = 1, \ldots, k+t$,

$$C_q(\tilde{f})\big(C_q(\tilde{j}_2)(c_u)\big) = C_q(\tilde{j}_2)\left(\sum_{v=1}^{k+t} c_v C_u^v\right),$$

because $C_q(\tilde{f})C_q(\tilde{j}_1) = C_q(\tilde{j}_1)C_q(\tilde{f}_1)$ and $C_q(\tilde{f})C_q(\tilde{j}_2) = C_q(\tilde{j}_2)C_q(\tilde{f}_2)$.

With an abuse of notation, let

$$j_{1*} : \mathbf{Z}\pi_1(X_1) \mapsto \mathbf{Z}\pi_1(X),$$
$$j_{2*} : \mathbf{Z}\pi_1(X_2) \mapsto \mathbf{Z}\pi_1(X),$$
$$i_{1*} : \mathbf{Z}\pi_1(A) \mapsto \mathbf{Z}\pi_1(X_1),$$
$$i_{2*} : \mathbf{Z}\pi_1(A) \mapsto \mathbf{Z}\pi_1(X_2)$$

be the induced ring homomorphisms, with base points

$$w(0),\ i_1\big(w(0)\big),\ i_2\big(w(0)\big) \text{ and } j_1 i_1\big(w(0)\big)$$

in the respective spaces. It is immediate to verify that for every $v = 1, \ldots, k+s$ and every $B \in \mathbf{Z}\pi_1(X_1)$ the equality $C_q(\tilde{j}_1)(b_v B) = (C_q(\tilde{j}_1)(b_v))j_{1*}(B)$ holds; obviously, the right action in the right hand side of the equation is thought out to be in $C_q(\widetilde{X})$. A similar equation holds true for $C_q(\tilde{j}_2)$. Therefore, for all $u = 1, \ldots, k+s$,

$$C_q(\tilde{f})\big(C_q(\tilde{j}_1)(b_u)\big) = \sum_{v=1}^{k+s} \big(C_q(\tilde{j}_1)(b_v)\big)j_{1*}(B_u^v)$$

and for all $u = 1, \ldots, k+t$,

$$C_q(\tilde{f})\big(C_q(\tilde{j}_2)(c_u)\big) = \sum_{v=1}^{k+t} \big(C_q(\tilde{j}_2)(c_v)\big)j_{2*}(C_u^v)$$

is verified. As a consequence, the trace

$$\operatorname{Tr}_{f_*}\big[C_q(\tilde{f})\big] = \sum_{u=1}^{k}\big[j_{1*}(B_u^u)\big] + \sum_{u=k+1}^{k+s}\big[j_{1*}(B_u^u)\big] + \sum_{u=k+1}^{k+t}\big[j_{2*}(C_u^u)\big],$$

where it is $j_{1*}(B_u^u) = j_{2*}(C_u^u)$ for every $u = 1, \ldots, k$. In fact, it is easy to see that for every $u, v = 1, \ldots, k$, the relations $B_v^u = i_{1*}(A_v^u)$ and $C_v^u = i_{2*}(A_v^u)$ are true. We are at last in the position of finishing the proof: because of the equalities

$$i_{1*}\big(\operatorname{Tr}_{f_{A*}}\big[C_q(\tilde{f}_A)\big]\big) = \sum_{u=1}^{k}\big[B_u^u\big],$$

$$i_{2*}\big(\operatorname{Tr}_{f_{A*}}\big[C_q(\tilde{f}_A)\big]\big) = \sum_{u=1}^{k}\big[C_u^u\big]$$

it is straightforward to show that for every $q \geqslant 0$

$$\operatorname{Tr}_{f_*}\big[C_q(\tilde{f})\big] + j_{1*}i_{1*}\operatorname{Tr} C_q(\tilde{f}_A) = j_{1*}\operatorname{Tr}_{f_{1*}}\big[C_q(\tilde{f}_1)\big] + j_{2*}\operatorname{Tr}_{f_{2*}}\big[C_q(\tilde{f}_2)\big],$$

and adding on the alternating sum, we obtain exactly the Pushout formula for this special case. □

Lemma 4.2. *The induced function $p_*: \mathcal{R}(f_2') \mapsto \mathcal{R}(f_2)$ is an index preserving bijection. Therefore, $\mathcal{L}(f_2) = p_*\mathcal{L}(f_2')$.*

Proof. We observe that we must take care of the base paths. Remember the above defined inclusion $\bar{\jmath}: X_2 \mapsto M(i_2)$ and the homotopy $H: \bar{\jmath}p \sim 1_{M(i_2)}$. By commutative property of $\mathcal{L}$ (cf. [1,4]), $\bar{\jmath}_*: \mathcal{R}(f_2, i_2(w)) \mapsto \mathcal{R}(\bar{\jmath}f_2p, \bar{\jmath}i_2(w))$ is an index preserving bijection. At the same time, H induces an index preserving bijection $H_*: \mathcal{R}(\bar{\jmath}f_2p, \bar{\jmath}i_2(w)) \mapsto \mathcal{R}(f_2', \bar{\jmath}i_2(w))$ because $f_2p = pf_2'$. Finally, take the path λ in $M(i_2)$ defined by $\lambda(s) := \bar{\imath}_2(w(0), s)$ for all $s \in I$; it induces a change of coordinates $\lambda_*: \mathcal{R}(f_2', \bar{\jmath}i_2(w)) \mapsto \mathcal{R}(f_2', ii_2(w))$. Therefore we can define an index preserving bijection $\lambda_*H_*\bar{\jmath}_*: \mathcal{R}(f_2, i_2(w)) \mapsto \mathcal{R}(f_2', ii_2(w))$, which takes every element $[\alpha] \in \mathcal{R}(f_2, i_2(w))$ into

$$[\lambda^{-1}\bar{\jmath}(\alpha)\bar{\jmath}i_2(w)f_2'(\lambda)ii_2(w^{-1})] \in \mathcal{R}(f_2', ii_2(w)).$$

Now we are going to prove that $p_*: \mathcal{R}(f_2', ii_2(w)) \mapsto \mathcal{R}(f_2, i_2(w))$ is the inverse of this bijection. For every element $[\alpha] \in \mathcal{R}(f_2, i_2(w))$, it is easy to see that

$$p_*\lambda_*H_*\bar{\jmath}_*([\alpha]) = [\alpha i_2(w)i_2(w^{-1})i_2(w)i_2(w^{-1})] = [\alpha].$$

On the other hand, take an element $[\beta] \in \mathcal{R}(f_2', ii_2(w))$. It is not difficult to find a homotopy relative to endpoints such that $\bar{\jmath}i_2(w)f_2'(\lambda)ii_2(w^{-1}) \sim \lambda$. Hence,

$$\lambda_*H_*\bar{\jmath}_*p_*([\beta]) = [\lambda^{-1}\bar{\jmath}p(\beta)\bar{\jmath}i_2(w)f_2'(\lambda)ii_2(w^{-1})]$$

is equal to $[\lambda^{-1}\bar{\jmath}p(\beta)\lambda]$. But if we take the map $K: I \times I \mapsto M(i_2)$ defined by $K(t,s) := H(\beta(t), s)$ for all $(t,s) \in I \times I$, we can modify it to have a homotopy $\lambda^{-1}\bar{\jmath}p(\beta)\lambda \sim \beta$. In conclusion, p_* is an index preserving bijection, because it is the inverse of $\lambda_*H_*\bar{\jmath}_*$. □

Lemma 4.3. *The induced function $\bar{p}_*: \mathcal{R}(f_1 \sqcup_{f_A} f_2') \mapsto \mathcal{R}(f)$ is an index preserving bijection. Therefore, $\mathcal{L}(f) = \bar{p}_*\mathcal{L}(f_1 \sqcup_{f_A} f_2')$.*

Proof. Let $M(i_1)$ be the mapping cylinder of the inclusion $i_1: A \mapsto X_1$; $M(i_1) = X_1 \times \{0\} \cup A \times I$, and let $ii_1: A \mapsto M(i_1)$ be the inclusion defined by $ii_1(a) := (a, 1)$ for all $a \in A$. Note that $X_1 \sqcup_A M(i_2) \approx M(i_1) \sqcup_A X_2$ in a natural way. We will identify these two spaces. Let $\bar{\imath}_2: M(i_1) \mapsto M(i_1) \sqcup_A X_2$ and $\imath\bar{\imath}_1: X_2 \mapsto M(i_1) \sqcup_A X_2$ be the maps induced by the pushout construction. We know that i_1 is a cofibration, so there exists a retraction $r: X_1 \times I \mapsto M(i_1)$. Let $L: X_1 \mapsto M(i_1)$ be the map defined by $L(x_1) := r(x_1, 1)$ for every $x_1 \in X_1$.

Let $\sigma: X \mapsto M(i_1) \sqcup_A X_2$ be the unique (by the universal property of pushouts) map such that $\sigma j_1 = \bar{\imath}_2 L$ and $\sigma j_2 = \imath\bar{\imath}_1$. We will show that σ is the homotopy inverse of $\bar{p}$. In any case, it induces an index preserving bijection

$$\sigma_*: \mathcal{R}(f\bar{p}\sigma, j_1i_1(w)) \mapsto \mathcal{R}(\sigma\bar{p}\ f_1 \sqcup_{f_A} f_2', \sigma j_1i_1(w))$$

because of the commutativity property and the equality $f\bar{p} = \bar{p}\ f_1 \sqcup_{f_A} f_2'$.

Define $H: X_1 \times I \mapsto X_1$ by setting $H(x_1, s) := \mathrm{pr}_1' r(x_1, s)$ for every $(x_1, s) \in X_1 \times I$, where $\mathrm{pr}_1' : M(i_1) \mapsto X_1$ is the projection over the first factor. By the universal property for $X \times I$, there exists a $K: X \times I \mapsto X$ such that $K(j_1 \times 1_I) = j_1 H$ and $K(j_2 \times 1_I) = j_2 \mathrm{pr}_1$, in which $j_1 \times 1_I : X_1 \times I \mapsto X \times I$ and $j_2 \times 1_I : X_2 \times I \mapsto X \times I$ are defined to be the identity on the second factor, and $\mathrm{pr}_1 : X_2 \times I \mapsto X_2$ is the projection on the first one. It turns out that $K(-,0) = 1_X$ and $K(-,1) = \bar{p}\sigma$ and that K is a homotopy relative to $j_2(X_2)$. This allows us to define an index preserving bijection $K_*: \mathcal{R}(f, j_1 i_1(w)) \mapsto \mathcal{R}(f\bar{p}\sigma, j_1 i_1(w))$ as in Section 2.2.

Let $C: M(i_1) \times I \mapsto M(i_1)$ be the map defined by $C((x_1,t),s) := r(x_1, (1-s)t + s)$ for every $(x_1, t) \in M(i_1) \subseteq X_1 \times I$ and for all $s \in I$, and $r : X_1 \times I \mapsto M(i_1)$ the retraction seen before. Let $\mathrm{pr}_1 : X_2 \times I \mapsto X_2$ be the projection over the first factor; let $\bar{\imath}_2 \times 1_I : M(i_1) \times I \mapsto (M(i_1) \sqcup_A X_2) \times I$ and $\imath\bar{\imath}_1 \times 1_I : X_2 \times I \mapsto (M(i_1) \sqcup_A X_2) \times I$ be the maps of the pushout diagram for $(M(i_1) \sqcup_A X_2) \times I$. Because of the universal property, there exists a unique $D: (M(i_1) \sqcup X_2) \times I \mapsto M(i_1) \sqcup X_2$ such that $D(\bar{\imath}_2 \times 1_I) = \bar{\imath}_2 C$ and that $D(\imath\bar{\imath}_1 \times 1_I) = \imath\bar{\imath}_1 \mathrm{pr}_1$. The following equations hold true:

$$D(-,0) = 1_{M(i_1) \sqcup_a X_2} \quad \text{and} \quad D(-,1) = \sigma\bar{p}.$$

This is because $C(-,0) = 1_{M(i_1)}$ and $C(-,1) = r(-,1) = Lp$. Let

$$D_*^{-1} : \mathcal{R}\big(\sigma\bar{p}\ f_1 \sqcup_{f_A} f_2', \sigma j_1 i_1(w)\big) \mapsto \mathcal{R}\big(f_1 \sqcup_{f_A} f_2', \sigma j_1 i_1(w)\big)$$

be the inverse of the index preserving bijection induced by D.

At last, let λ be the path in $M(i_1) \sqcup_A X_2$ defined by $\lambda(s) := \bar{\imath}_2(i_1(w(0)), 1-s)$ for all $s \in I$. Let $\lambda_* : \mathcal{R}(f_1 \sqcup_{f_A} f_2', \sigma j_1 i_1(w)) \mapsto \mathcal{R}(f_1 \sqcup_{f_A} f_2', \imath\bar{\imath}_2 i_1(w))$ be the change of coordinates induced by λ.

Consider the index preserving bijection given by the composition

$$\lambda_* D_*^{-1} \sigma_* K_*.$$

We wish to prove that $\lambda_* D_*^{-1} \sigma_* K_*$ is precisely the inverse of $\bar{p}_*$. We first observe that it takes the element $[\alpha]$ of $\mathcal{R}(f, j_1 i_1(w))$ to

$$\big[\lambda^{-1}\sigma(\alpha)\sigma j_1 i_1(w)\ f_1 \sqcup_{f_A} f_2'\ \imath\bar{\imath}_2 i_1\big(w^{-1}\big)\big]$$

of $\mathcal{R}(f_1 \sqcup_{f_A} f_2', \imath\bar{\imath}_2 i_1(w))$. Next, we can easily find a homotopy to show that the path $\sigma j_1 i_1(w)\ f_1 \sqcup_{f_A} f_2'(\lambda)\ \imath\bar{\imath}_2 i_1(w^{-1})$ is homotopic rel. endpoints to λ. Therefore, $\lambda_* D_*^{-1} \sigma_* K_*([\alpha]) = [\lambda^{-1}\sigma(\alpha)\lambda]$. Now

$$\bar{p}_*\ \lambda_* D_*^{-1} \sigma_* K_*\big([\alpha]\big) = \big[\bar{p}\sigma(\alpha)\big] = [\alpha]$$

because $K : 1_X \sim \bar{p}\sigma\ \mathrm{rel}\, j_2(X_2)$. On the other hand, take $[\beta] \in \mathcal{R}(f_1 \sqcup_{f_A} f_2', \imath\bar{\imath}_2 i_1(w))$. We see that $\lambda_* D_*^{-1} \sigma_* K_* \bar{p}_*([\beta]) = [\lambda^{-1}\sigma\bar{p}(\beta)\lambda]$. Define the map $A(t,s) := D(\beta(t), s)$ for all $(t,s) \in I \times I$. We obtain a map $A: I \times I \mapsto M(i_1) \sqcup_A X_2$ such that $A(t,0) = \beta(t)$, $A(t,1) = \sigma\bar{p}(\beta(t))$, and $A(0,s) = A(1,s) = \lambda^{-1}(s)$ for all $t, s \in I$. Hence $\beta \sim \lambda^{-1}\sigma\bar{p}(\beta)\lambda$, and $\lambda_* D_*^{-1} \sigma_* K_* \bar{p}_*([\beta]) = [\beta]$. Therefore $\bar{p}_*$ is an index preserving bijection. $\square$

We are now in a position to prove the Pushout formula. Because $i\bar{\imath}_2$ and i_1 are cellular inclusions, Lemma 4.1 holds true for $f_1 \sqcup_{f_A} f_2'$. Hence,

$$\mathcal{L}(f_1 \sqcup_{f_A} f_2') = \imath\bar{\imath}_{2*}\mathcal{L}(f_1) + \bar{\imath}_{1*}\mathcal{L}(f_2') - \imath\bar{\imath}_{2*}i_{1*}\mathcal{L}(f_A)$$

with the choice of base paths mentioned before. But, using Lemma 4.3, we obtain

$$\mathcal{L}(f) = \bar{p}_*\big(\imath\bar{\imath}_{2*}\mathcal{L}(f_1) + \bar{\imath}_{1*}\mathcal{L}(f_2') - \imath\bar{\imath}_{2*}i_{1*}\mathcal{L}(f_A)\big)$$

and hence, due to the fact that $\bar{p}_*\imath\bar{\imath}_{2*} = j_{1*}$, $\bar{p}_*\bar{\imath}_{1*} = j_{2*}p_*$ and $\bar{p}_*i\bar{\imath}_{2*}i_{1*} = j_{1*}i_{1*}$, the equation $\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}p_*\mathcal{L}(f_2') - j_{1*}i_{1*}\mathcal{L}(f_A)$. Therefore, by Lemma 4.2, the Pushout formula holds true.

5. Examples

5.1. The converse of the Lefschetz fixed point theorem

The main tool for defining maps on surfaces is the Dehn twist: for each $k \in \mathbf{Z}$ define the map $T_k : S^1 \times I \mapsto S^1 \times I$ by $T_k(\mathrm{e}^{2\pi it}, s) := (\mathrm{e}^{2\pi i(t+ks)}, s)$ for each $t, s \in I$. It is easy to see that T_k is the identity if restricted to the boundary $\partial(S^1 \times I) = S^1 \times \partial I$. Let X_1 be the surface obtained by removing a small open disc D^2, which does not intersect $a := S^1 \times \{\mathrm{e}^0\}$ and $b := \{\mathrm{e}^0\} \times S^1$, from a torus surface $S^1 \times S^1$. We may suppose the closure $\overline{D}^2$ of D^2 to intersect a and b only in $x_0 := (\mathrm{e}^0, \mathrm{e}^0)$. Moreover, we may endow X_1 with a cellular structure such that ∂X_1 is a subcomplex of X_1, and such that the we could take suitable cellular approximates of each of the following maps. Let a and b be the free generators of the fundamental group of X_1. If we take tubular neighborhoods of a and b, we can define Dehn twists over the whole surface. For all $h \in \mathbf{Z}$ we can define φ_h on X_1 such that $\varphi_h(a) \sim ab^h$ and $\varphi_h(b) = b$, if we twist around b. Similarly, for each $k \in \mathbf{Z}$, we define φ_k' such that $\varphi_k'(a) = a$ and $\varphi_k'(b) \sim ba^k$, if we twist around a. Both φ and φ' are the identity if restricted to ∂X_1. We use a and b to denote either the paths in X_1 or the corresponding elements in $\pi_1(X_1, X_0)$. By composition, define $f_k : X_1 \mapsto X_1$ by $f_k := \varphi_k'\varphi_{-1}$ with $k \geqslant 2$. It is the identity on ∂X_1, and $f_k(a) = a^{1-k}b^{-1}$, $f_k(b) = ba^k$. Let the base path for f_k be the constant path at x_0. It is easy to find a retraction $r : X_1 \mapsto S^1 \bigvee S^1 = a \bigvee b$, which is a deformation. By using commutativity and homotopy invariance, we can show that there exists an index preserving bijection $i_* : \mathcal{R}(rf_ki) \mapsto \mathcal{R}(f_k)$ defined in the natural way. Therefore we need to compute $\mathcal{L}(rf_ki)$, as a self map of $S^1 \bigvee S^1$. Using Fox calculus (in a different way of the original one), remembering that $k \geqslant 2$, after some computing we obtain that

$$\mathcal{L}(rf_ki) = \sum_{j=1}^{k-1}[a^j] \in \mathbf{Z}\mathcal{R}(rf_ki)$$

as in [2]. Therefore,

$$\mathcal{L}(f_k) = \sum_{j=1}^{k-1} i_*[a^j] = \sum_{j=1}^{k-1}[a^j] \in \mathbf{Z}\mathcal{R}(f_k)$$

by an abuse of notation.

Let now X_2 be any finite connected CW-complex, and $i_2 : \partial X_1 \mapsto X_2$ be any cellular map. Let $\chi(X_2)$ denote the Euler–Poincaré characteristic of X_2. Suppose that $\chi(X_2) \leqslant -1$. Let $k := 1 - \chi(X_2) \geqslant 2$. Attach X_1 to X_2 via i_2, and let X be the space obtained. This is a pushout construction. Define a self map $f : X \mapsto X$ by extending f_k over the whole X by the identity on X_2. The Pushout formula here gives

$$\mathcal{L}(f) = \sum_{j=1}^{k-1} j_{1*}[a^j] + \chi(X_2)[1] - 0[1],$$

because $\mathcal{L}(1_Y) = \chi(Y)[1]$ for any space Y.

Now we are going to show that $\mathcal{L}(f) \neq 0$ and $L(f) = 0$. By the generalized Seifert–Van Kampen theorem, $\pi_1(X)$ is the amalgamated direct product of $\pi_1(X_1)$ and $\pi_1(X_2)$ via the homomorphisms $i_{1\pi} : \pi_1(\partial X_1) \mapsto \pi_1(X_1)$ and $i_{2\pi} : \pi_1(\partial X_1) \mapsto \pi_1(X_2)$. Therefore, by the universal property, there exists a unique homomorphism $q : \pi_1(X) \mapsto \mathbf{Z} \oplus \mathbf{Z}$ that extends the abelianization homomorphism $A : \pi_1(X_1) \mapsto \mathbf{Z} \oplus \mathbf{Z}$ and the trivial one $T : \pi_1(X_2) \mapsto (0,0) \in \mathbf{Z} \oplus \mathbf{Z}$. Let H be the Kernel of such homomorphism. Then H is a normal subgroup of $\pi_1(X)$ such that $f_*(H) \subseteq H$. Therefore we can take the relative Reidemeister classes $\mathcal{R}(f_k, H)$ of f_k (cf. [4]), namely the set of Reidemeister orbits in $\pi_1(X)/H$; we obtain the abelian group $\mathbf{Z} \oplus \mathbf{Z}$ and hence it is rather immediate to distinguish classes in it. Using these arguments we can show that $[1], [a], [a^2], \ldots, [a^{k-1}]$ are k distinct elements of $\mathcal{R}(f_k)$, for each $k \geqslant 2$. Hence $\mathcal{L}(f_k) \neq 0$. Moreover, $L(f_k) = 0$ by the choice of k, and $N(f_k) = k$. Note that if $k = 2$ and X_2 is a torus minus a disc, this example reduces to the classical one by McCord [5]. Incidentally, by the existence of such maps, we have proved the following proposition:

Proposition 5.1.1. *The converse of the Lefschetz fixed point theorem does not hold true for any space obtained by attaching cellularly the boundary of a torus minus a disc to a space with negative characteristic* $\chi \leqslant -1$.

This happens to all the orientable surfaces with negative characteristic, and to all but finitely many nonorientable surfaces. So it is also proved that, among surfaces, the converse of the Lefschetz theorem holds true only for a finite set of them. For more information, the reader should consult [1].

5.2. *N is far from L*

Let X_1 be the orientable surface of genus 2, that is, the connected sum of two tori minus an open small disc D^2. Again, as in the previous example, we can take simple paths a_1, b_1, a_2, b_2 which are free generators of $\pi_1(X_1, x_0)$, where x_0 is a point of the boundary $\partial X_1 \approx S^1$. Next, we take tubular neighborhoods of a_1, a_2, b_1 and b_2 and for given integers $k_1 \geqslant 2, k_2 \geqslant 1$, we take suitable Dehn twists; define a cellular map $f_1 : X_1 \mapsto X_1$ such that $f_1(a_1) = a_1^{1-k_1} b_1^{-1}$, $f_1(b_1) = b_1 a_1^{k_1}$, $f_1(a_2) = a_2 b_2 a_2^{k_2}$, $f_1(b_2) = b_2 a_2^{k_2}$ and $f|_{\partial X_1}$ is the identity on ∂X_1. As before, using a deformation retraction and Fox calculus, we can show after a little computation that

$$\mathcal{L}(f_1) = \sum_{j=1}^{k_1-1} [a_1^j] - \sum_{i=0}^{k_2-1} [a_2^{-i} b_2^{-1} a_2^{-1}] - 2 \cdot [1].$$

Now, let X_2 be a space with characteristic $\chi(X_2)$. It turns out that $\mathcal{L}(1_{X_2}) = \chi(X_2)[1]$. Let X be the space obtained by attaching X_1 to X_2 via a cellular map $i_2 : \partial X_1 \mapsto X_2$. Define a self map f on X by extending f_1 on the whole X with the identity on X_2. Take the constant path at x_0 as base path for all self maps. Applying the Pushout formula, we obtain that $\mathcal{L}(f) = j_{1*}\mathcal{L}(f_1) + j_{2*}\chi(X_2)[1]$, because $\mathcal{L}(1_{\partial X_2}) = 0$. Therefore,

$$\mathcal{L}(f) = \sum_{j=1}^{k_1-1} j_{1*}[a_1^j] - \sum_{i=0}^{k_2-1} j_{1*}[a_2^{-i} b_2^{-1} a_2^{-1}] + (\chi(X_2) - 2) \cdot [1].$$

To compute the Nielsen number $N(f)$ we need to detect which addends are distinct in $\mathcal{R}(f)$. As in the previous example, take the smallest normal subgroup H of $\pi_1(X)$ containing $j_{2\pi}(\pi_1(X_2))$ and the commutators of $j_{1\pi}(\pi_1(X_1))$. By the universal property for $\pi_1(X)$ (Seifert–Van Kampen), we can actually define H as the kernel of the unique epimorphism $\rho : \pi_1(X) \mapsto \mathbf{Z}^4$ such that $\rho j_{1\pi}$ is the abelianization of $\pi_1(X_1)$ over $\mathbf{Z}^4$ and $\rho j_{2\pi}$ is the constant morphism on the zero of $\mathbf{Z}^4$. Let us operate in the relative level, and compute $N(f;H)$ and $\mathcal{L}(f;H)$. Clearly, $\pi_1(X)/H = \mathbf{Z}^4 = (a_1, b_1, a_2, b_2)$. Note that

$$\mathcal{L}(f;H) = \sum_{j=1}^{k_1-1} j_{1*}[a_1^j] - \sum_{i=0}^{k_2-1} j_{1*}[a_2^{-i-1}] + (\chi(X_2) - 2) \cdot [1],$$

because $\mathrm{Im}(1 - \bar{f}_*) = k_1\mathbf{Z} \oplus \mathbf{Z} \oplus k_2\mathbf{Z} \oplus \mathbf{Z} \subseteq \mathbf{Z}^4$ and so $a_2^{-i-1} b_2^{-1}$ is in the same relative Reidemeister orbit as a_2^{-i-1}. By the same argument, we can show that

$$\mathcal{L}(f;H) = \sum_{j=1}^{k_1-1} j_{1*}[a_1^j] - \sum_{i=1}^{k_2-1} j_{1*}[a_2^i] + (\chi(X_2) - 3) \cdot [1]$$

and hence $N(f;H) = k_1 + k_2 - 1$ if $\chi(X_2) \neq 3$, $N(f;H) = k_1 + k_2 - 2$ if $\chi(X_2) = 3$. On the other side, $L(f) = k_1 - k_2 + \chi(X_2) - 3$. What about $\mathcal{L}(f)$ and $N(f)$? The only term that could cancel out some other term is $[a_2^{-k_2+1} b_2^{-1} a_2^{-1}]$, which collapses into $[1]$ at the relative level. But the equality $a_2^{-k_2+1} b_2^{-1} a_2^{-1} = a_2 f(a_2^{-1})$ implies that $[a_2^{-k_2+1} b_2^{-1} a_2^{-1}] = [1]$, so that $N(f) = N(f;H)$.

Proposition 5.2.1. *Let X be a space obtained by attaching cellularly the boundary of the orientable surface of genus 2 minus a 2-disc to a finite connected CW-complex. Then, given arbitrary integers $L \in \mathbf{Z}$ and $N \geqslant 2$, it is possible to find a self map f of X such that $L(f) = L$ and $N(f) \geqslant N$.*

Proof. It is enough to take k_2 such that $k_2 \geqslant 1$, $k_2 \geqslant \chi(X_2) - L - 1$ and $k_2 \geqslant \frac{1}{2}(N - L + \chi(X_2) - 1)$. Let $k_1 := k_2 + L + 3 - \chi(X_2)$. Then $k_1 \geqslant \chi(X_2) - L - 1 + L + 3 - \chi(X_2) = 2$ and $k_2 \geqslant 1$, showing the existence of a self map f such that $L(f) = k_1 - k_2 + \chi(X_2) - 3$.

Indeed, $L(f) = L + 3 - \chi(X_2) + \chi(X_2) - 3 = L$. Moreover, if either $\chi(X_2) = 0$ or $\chi(X_2) \neq 0$, it turns out that

$$N(f) \geqslant k_1 + k_2 - 2 = k_2 + L + 3 - \chi(X_2) + k_2 - 2$$

and then

$$N(f) \geqslant N - L + \chi(X_2) - 1 + L + 1 - \chi(X_2) = N. \qquad \square$$

Given a space X, we say that N is *far from* an integer L whenever there exist maps f such that $L(f) = L$ and $N(f) \gg |L|$. For almost all the surfaces the hypotheses of the previous proposition are true, so for almost all surfaces N is far from L.

5.3. Higher dimensions

Let X be a surface, and for any integer $d \geqslant 3$ consider the d-manifold $X \times S^{d-2}$. For every self map f of X, let $\phi : S^{d-2} \mapsto S^{d-2}$ be the constant map, with $N(\phi) = 1 = L(\phi)$; there exists a self map $\varphi : X \times S^{d-2} \mapsto X \times S^{d-2}$, defined by $\varphi := f \times \phi$. The map φ has the property that $N(\varphi) = N(f)$ and $L(\varphi) = L(f)$. Therefore the foregoing examples may extend to higher dimensions. There are infinitely many manifolds of each dimension for which N is far from L. Consequently, there are infinitely many manifolds of each dimension for which the converse of the Lefschetz fixed point theorem does not hold true.

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